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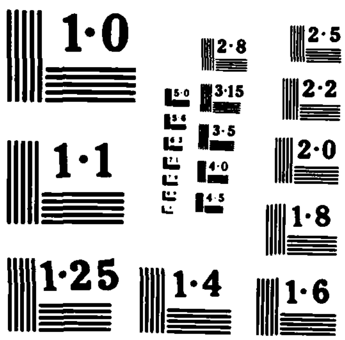


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SYMMETRIC STOCHASTIC PETRI NETS

by

Lindsay A. Prisgrove and Gerald S. Shedler

TECHNICAL REPORT NO. 7

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## 1. INTRODUCTION

The stochastic Petri net (SPN) model is well suited to formal representation of concurrency, synchronization, and communication (cf., Marsan, Conte, and Balbo [11], Molloy [14,15], Natkin [16], Symons [18]). An SPN is specified by a finite set of *places* and a finite number of *transitions* along with a *normal input function*, an *inhibitor input function*, and an *output function* (each of which associates a set of places with a transition). A *marking* of an SPN is an assignment of zero or more *tokens* to the places in the net. A transition is *enabled* whenever there is at least one token in each of its normal input places and no tokens in any of its inhibitor input places; otherwise, it is *disabled*. A transition *fires* by removing one token (per place) from a random subset of its normal input places and depositing one token (per place) in a random subset of its output places.

Heuristically, an SPN changes marking in accordance with the firing of a transition enabled in the current marking. Each of the transitions enabled in a marking competes to change the marking and each of these enabled transitions has its own stochastic mechanism for determining the next marking. At each firing of a transition in the SPN, new transitions may become enabled. For each of these new enabled transitions, a clock indicating the time until the transition fires is set according to an independent stochastic mechanism. (There is no restriction to exponentially distributed transition firing times.) If an enabled transition does not trigger a marking change but is enabled in the next marking, its clock continues to run; if such a transition is not enabled in the next marking, its clock reading is abandoned.

The SPN representation provides a means of generating sample paths for the underlying stochastic process of a discrete event simulation. This representation is particularly useful in connection with non-Markovian systems. The "state of the system at time $t$" defines the underlying stochastic process of a discrete event simulation. When the current state of the stochastic process is $s$ and a previously scheduled event, $e^*$, occurs, the process moves to a new state, $s'$. The marking of an SPN corresponds to the state of the process and the firing of a transition corresponds to the occurrence of an event. The graphical representation (bipartite graph of places and transitions) of an SPN is particularly useful in that it incorporates a considerable amount of information about the set of events that can occur when the process is in state $s$ and the sets of "new events" and "old events" when event $e^*$ triggers a transition from state $s$ to state $s'$.

Although steady state estimation for an arbitrary SPN is a very difficult problem, Haas and Shedler [4] have provided estimation procedures for SPN's that are regenerative processes. To establish the regenerative property for an SPN, it is necessary to show the existence of an infinite sequence of random time points at which the process probabilistically restarts. It is often clear that an SPN probabilistically restarts when a particular transition fires leaving the system with a fixed marking. For specific models, however, it is nontrivial to determine conditions (distributional assumptions) under which this occurs infinitely often with probability one. Using recurrence theory (Haas and Shedler [3]) for generalized semi-Markov processes (König, Matthes, and Nawrotzki [8,9], Matthes [12], Whitt [19]), conditions are given in [4] which ensure that an SPN is a regenerative process in continuous time with finite expected time between regeneration points.

In this paper we focus on SPN's with special structure and define a symmetric SPN. Informally, an SPN is symmetric if there are mappings of places onto places, markings onto markings, and transitions onto transitions which preserve the sets of enabled transitions, the new marking probabilities, the sets of new transitions, and the clock setting distributions. Symmetric SPN's have application to representation of ring networks with equally spaced, identical ports; cf. Loucks, Hamacher, and Preiss [10].

Section 2 provides the formal definition of an SPN given in [4] along with conditions which ensure that an SPN is a regenerative process and that the expected time between regeneration points is finite. Using a geometric trials recurrence criterion (Iglehart and Shedler [6]), Proposition (2.19) postulates the existence of a transition, $e^*$, and a marking, $s_0'$, such that transition $e^*$ fires and the new marking is $s_0'$ infinitely often with probability one. Conditions on the old clocks ensure that the process probabilistically restarts at these times. This result is the basis for regenerative simulation of SPN's.

Section 3 considers the steady state estimation problem for symmetric SPN's. Under the assumptions of Proposition (2.19), regenerative cycles defined by the times at which transition $e^*$ fires and the new marking is $s_0'$ can be decomposed into independent, nonidentically distributed blocks. We show that point estimates and confidence intervals for characteristics of symmetric functions of the limiting distribution can be obtained by simulating the symmetric SPN in blocks.

In Section 4 we develop estimation procedures for passage times in the SPN setting. Formal specification of a sequence $\{P_n':n \geq 1\}$ of passage times in a symmetric SPN $\{X(t):t \geq 0\}$ with marking set, $S$, and transition set, $E$, is in terms of four subsets ($A_1$, $A_2$, $B_1$, and $B_2$) of $S$. The sets $B_1$ and $B_2$ define the random times $\{T_j':j \geq 1\}$ at

which a passage time terminates. (The sets $A_1$ and $A_2$ define the random times at which a passage time starts.) Proposition (4.1) postulates the existence of $e^* \in E$ and $s_0, s_0' \in S$ such that transition $e^*$ fires and the marking changes from $s_0$ to $s_0'$ infinitely often with probability one and these transition firing times correspond to termination of a passage time with no other passage times underway. Conditions on the "old clocks" ensure that $\{(X(T_n'), P_{n+1}'):n \geq 0\}$ is a regenerative process in discrete time and that the expected time between regeneration points is finite.

Section 4 provides two estimation procedures for passage times in a symmetric SPN. Each of these procedures rests on the assumption that there exist $e^* \in E$, $s_0 \in B_1$, and $s_0' \in B_2'$ satisfying the conditions of Proposition (4.1). The regenerative structure guarantees that $P_n' \Rightarrow P$ as $n \rightarrow \infty$ and the goal of the simulation is the estimation of $r(f) = E\{f(P)\}$, where $f$ is a real-valued measurable function.

Estimates for $r(f)$ can be based on measurement of passage times $\{P_n^1:n \geq 1\}$ (a particular random subsequence of $\{P_n':n \geq 1\}$) and simulation of the underlying SPN in regenerative cycles. Alternatively, exploiting properties of a symmetric SPN, estimates can be based on measurement of passage times $\{P_n':n \geq 1\}$ and simulation of the underlying SPN in independent, nonidentically distributed blocks. This estimation procedure extracts more passage time information from a simulation of fixed length and should provide estimates for $r(f)$ that are relatively more accurate. In Section 5 we verify that this is indeed the case by showing that the resulting confidence intervals are shorter.

## 2. REGENERATIVE STOCHASTIC PETRI NETS

Following Haas and Shedler [4], formal definition of an SPN is in terms of a general state space Markov chain (GSSMC) which describes the process at successive epochs of transition firing. Let $D = \{1,2,...,L\}$ be the index set for a finite collection of *places* and let $E = \{e_1,e_2,...,e_M\}$ be a finite set of *transitions*. Denote by $S$ the finite or countable set of *markings* and for $s \in S$ write $s = (s_1,s_2,...,s_L)$, where $s_j$ is the number of tokens in place $j$, $j \in D$. Denote the index set of the *normal input places* for transition $e \in E$ by $I(e) \subseteq D$, the index set of the *inhibitor input places* by $L(e) \subseteq D$, and the index set of the *output places* by $J(e) \subseteq D$. We assume that

$$L(e) \bigcap I(e) = \emptyset$$

for all $e \in E$. For $s = (s_1,s_2,...,s_L) \in S$, set

$$E(s) = \left\{e \in E\colon s_j \geq 1 \text{ for } j \in I(e) \text{ and } s_j = 0 \text{ for } j \in L(e)\right\} \tag{2.1}$$

so that $E(s)$ is the set of transitions that are enabled when the marking of the SPN is $s$. When the marking of the SPN is $s$ the firing of an enabled transition $e \in E(s)$ triggers a marking change to $s'$. We denote by $p(s';s,e)$ the probability that the new marking is $s'$ given that transition $e$ fires when the marking is $s$. For all $s = (s_1,s_2,...,s_L)$, $s' = (s'_1,s'_2,...,s'_L) \in S$, and $e \in E(s)$ we assume that $p(s';s,e) > 0$ only if

(i) $s_j - 1 \leq s'_j \leq s_j$ for all $j \in I(e) \cap (D - J(e))$,

(ii) $s_j - 1 \leq s'_j \leq s_j + 1$ for all $j \in I(e) \cap J(e)$,

(iii) $s_j \leq s'_j \leq s_j + 1$ for all $j \in J(e) \cap (D - I(e))$, and

(iv) $s'_j = s_j$ for all $j \in (D - J(e) - I(e))$.

The actual enabled transition $e$ which triggers a marking change when the marking is $s$ depends on *clocks* associated with the enabled transitions and the *speeds* at which these

clocks run. Each such clock records the remaining time until the transition fires. We denote by $r_{si}$ ( $\geq 0$) the deterministic rate at which the clock associated with transition $e_i$ runs when the marking is $s$; for each $s \in S$, $r_{si} = 0$ if $e_i \notin E(s)$. We assume that $r_{si} > 0$ for some $e_i \in E(s)$. (Typically in applications, all speeds $r_{si}$ are equal to one. There are, however, models in which speeds other than unity as well as state-dependent speeds are convenient.)

For $s \in S$ define $C(s)$ to be the set of possible clock readings when the marking is $s$:

$$C(s) = \Big\{(c_1,\ldots,c_M)\colon c_i \geq 0 \text{ and } c_i > 0 \text{ if and only if } e_i \in E(s);$$

(2.2) $$c_i r_{si}^{-1} \neq c_j r_{sj}^{-1} \text{ for } i \neq j \text{ with } c_i c_j r_{si} r_{sj} > 0\Big\}.$$

The conditions in Equation (2.2) ensure that no two transitions fire simultaneously as defined below. The clock with reading $c_i$ is said to be *active* when the marking is $s$ if transition $e_i$ is enabled ($e_i \in E(s)$). For $s \in S$ and $c \in C(s)$, let

(2.3) $$t^* = t^*(s,c) = \min_{\{i: e_i \in E(s)\}} \{c_i r_{si}^{-1}\},$$

where $c_i r_{si}^{-1}$ is taken to be $+\infty$ when $r_{si} = 0$. Also set

(2.4) $$c_i^* = c_i^*(s,c) = c_i - t^*(s,c) r_{si},\ e_i \in E(s)$$

and

(2.5) $$i^* = i^*(s,c) = i \text{ such that } e_i \in E(s) \text{ and } c_i^*(s,c) = 0.$$

Beginning with marking $s$ and clock vector $c$, $t^*(s,c)$ is the time to the next transition firing and $i^*(s,c)$ is the index of the unique firing transition $e^* = e^*(s,c) = e_{i^*(s,c)}$.

At a marking change from $s$ to $s'$ triggered when transition $e^*$ fires, new clock times are generated for each $e' \in N(s';s,e^*) = E(s') - (E(s) - \{e^*\})$. The distribution function of such a new clock time is denoted by $F(\cdot;s',e',s,e^*)$ and we assume that $F(0;s',e',s,e^*) = 0$. For $e' \in O(s';s,e^*) = E(s') \cap (E(s) - \{e^*\})$, the old clock reading is kept after $e^*$ fires. For $e' \in (E(s) - \{e^*\}) - E(s')$, transition $e'$ (which was enabled before transition $e^*$ fired) is disabled.

Next consider a GSSMC $\{(S_n,C_n):n \geq 0\}$ having state space

$$\Sigma = \bigcup_{s \in S} (\{s\} \times C(s))$$

and representing the marking ($S_n$) and vector ($C_n$) of clock readings at successive transition firing times. (The $i$th coordinate of the vector $C_n$ is denoted by $C_{n,i}$.) The transition kernel of the Markov chain $\{(S_n,C_n):n \geq 0\}$ is

$$\text{(2.6)} \qquad P((s,c),A) = p(s';s,e^*) \prod_{e_i \in N(s')} F(a_i;s',e_i,s,e^*) \prod_{e_i \in O(s')} 1_{[0,a_i]}(c_i^*),$$

where $N(s') = N(s';s,e^*)$, $O(s') = O(s';s,e^*)$, and

$$A = \{s'\} \times \left\{(c_1',\ldots,c_M') \in C(s') \colon c_i' \leq a_i \text{ for } e_i \in E(s')\right\}.$$

The set $A$ is the subset of $\Sigma$ which corresponds to the SPN changing marking to $s'$ with the reading $c_i'$ on the clock associated with transition $e_i \in E(s')$ set to a value in $[0,a_i]$. (We suppose that the clock setting distributions are such that $P((s,c),\Sigma) = 1$.)

Finally, the SPN is a piecewise constant continuous time process constructed from the GSSMC $\{(S_n,C_n):n \geq 0\}$ in the following manner. Set $\zeta_0 = 0$ and denote by $\zeta_n$ the time

of the $n$th transition firing, $n \geq 0$. (We assume that

$$P\{\sup_{n \geq 1} \zeta_n = +\infty \mid (S_0, C_0)\} = 1 \text{ a.s.}$$

for all initial states $(S_0, C_0)$.) Then set

$$X(t) = S_{N(t)}, \tag{2.7}$$

where

$$N(t) = \max\{n \geq 0 : \zeta_n \leq t\}. \tag{2.8}$$

The process $\{X(t): t \geq 0\}$ defined by Equation (2.7) is an SPN.

Example (2.9) illustrates the use of the SPN framework for formal specification of a discrete-event simulation. For ring networks with $N$ ports, reference to port index "$j$" is to be interpreted as reference to index $j - 1 \pmod N + 1$. In the graphical representation of an SPN, places are drawn as circles and transitions as bars. Directed arcs connect transitions to output places and input places to transitions. Tokens are drawn as black dots.

(2.9) EXAMPLE (Token ring 1). Consider a unidirectional ring network having a fixed number of *ports*, labelled $1,2,\ldots,N$ in the direction of signal propagation; see Figure 1. At each port *message packets* arrive according to a random process. A single *control token* (denoted by $T$ in Figure 1) circulates around the ring from one port to the next. The time for the token to propagate from port $j-1$ to port $j$ is a positive constant, $R_{j-1}$, $j = 1,2,\ldots,N$. When a port observes the token and there is a packet queued for transmission the port converts the token to a *connector* ($C$) and transmits a packet followed by the token pattern; the token continues to propagate if there is no packet

queued for transmission. By destroying the connector prefix the port removes the transmitted packet when it returns around the ring.

Assume that the time for port $j$ to transmit a packet is a positive random variable, $L_j$, with finite mean. Also assume that packets arrive at individual ports randomly and independently of each other: the time from end of transmission by port $j$ until the arrival of the next packet for transmission by port $j$ is a positive random variable, $A_j$, with finite mean. Note that there is at most one packet queued for transmission at any time at any particular port.

Following [4], take $D = \{1,2,\ldots,4N\}$ and $E = \{e_1,e_2,\ldots,e_{3N}\}$. (See Figure 2 for $N = 2$.) Set

(2.10) $$L(e_{3j-2}) = L(e_{3j-1}) = L(e_{3j}) = \emptyset,$$

(2.11) $$I(e_{3j-2}) = \{4j-2\},\ I(e_{3j-1}) = \{4j-3,\ 4j-1\},\ I(e_{3j}) = \{4(j-1)\},$$

and

(2.12) $$J(e_{3j-2}) = \{4j-3\},\ J(e_{3j-1}) = \{4j-2,\ 4j\},\ J(e_{3j}) = \{4j-1,\ 4j\},$$

$j = 1,2,\ldots,N.$

The transitions have the following interpretation: $e_{3j-2}$ = "arrival of packet for transmission by port $j$," $e_{3j-1}$ = "end of transmission by port $j$," and $e_{3j}$ = "observation of token by port $j$," $j = 1,2,\ldots,N$. The interpretation of the places is as follows. Let $t \geq 0$ and $s = (s_1,s_2,\ldots,s_{4N}) \in S$. If the marking of the SPN at time $t$ is $s$ then $s_{4j-3} = 1$ if and only if at time $t$ port $j$ is transmitting a packet or there is a packet waiting for transmission by port $j$; $s_{4j-2} = 1$ if and only if at time $t$ port $j$ is not transmitting a packet

and there is no packet waiting for transmission by port $j$; $s_{4j-1} = 1$ if and only if at time $t$ port $j$ is transmitting a packet; and $s_{4j} = 1$ if and only if at time $t$ the control token is propagating to port $j+1$, $j = 1,2,\ldots,N$. (Otherwise $s_j = 0$, $j = 1,2,\ldots,4N$.)

Set

$$S' = \{(s_1,s_2,\ldots,s_{4N}) : s_j = 0 \text{ or } 1 \text{ for } 1 \le j \le 4N;\ s_1 + s_2 + \ldots + s_{4N} = N + 1\}.$$

The set, $S$, of markings is

$$S = \Big\{(s_1,s_2,\ldots,s_{4N}) \in S' : s_{4j-3} + s_{4j-2} = 1 \text{ and } s_{4j-2}s_{4j-1} = 0 \text{ for } 1 \le j \le N\Big\}. \tag{2.13}$$

(It follows that $|S| = 3N2^{N-1}$. In any marking there are exactly $N+1$ tokens. There is at most one token in each place. Each of the disjoint sets of places $\{4j-3, 4j-2\}$ contains exactly one token indicating whether or not port $j$ has a packet queued for transmission. The set of places $D - \{1,2,5,6,\ldots,4N-3,4N-2\}$ contains exactly one token indicating the position and status of the control token. There can never be tokens at places $4j-2$ and $4j-1$ simultaneously, reflecting the fact that there can be no arrival of a packet for transmission by port $j$ during a transmission by port $j$.)

The new marking probabilities are as follows. If $e = e_{3j-2} =$ "arrival of packet for transmission by port $j$," then $p(s';s,e) = 1$ when

$$s = (s_1,\ldots,s_{4(j-1)},0,1,0,s_{4j},s_{4j+1},\ldots,s_{4N}) \in S \text{ and } s' = (s_1,\ldots,s_{4(j-1)},1,0,0,s_{4j},s_{4j+1},\ldots,s_{4N}).$$

If $e = e_{3j-1} =$ "end of transmission by port $j$," then $p(s';s,e) = 1$ when

$$s = (s_1,\ldots,s_{4(j-1)},1,0,1,0,s_{4j+1},\ldots,s_{4N}) \in S \text{ and } s' = (s_1,\ldots,s_{4(j-1)},0,1,0,1,s_{4j+1},\ldots,s_{4N}).$$

If $e = e_{3j} =$ "observation of token by port $j$," then $p(s';s,e) = 1$ when

$$s = (s_1,\ldots,s_{4(j-1)-1},1,1,0,0,0,s_{4j+1},\ldots,\ldots,s_{4N}) \in S \text{ and } s' = (s_1,\ldots,s_{4(j-1)-1},0,1,0,1,0,s_{4j+1},\ldots,s_{4N})$$

and

$$T_j(l) = \inf \left\{ \zeta_n > S_{j-1}(l) \colon X(\zeta_n) \in B_2^l,\ X(\zeta_{n-1}) \in B_1^l \right\},\ j \geq 1.$$

Note that since the SPN is symmetric the sets $A_1^l$, $A_2^l$, $B_1^l$, and $B_2^l$ satisfy the conditions which ensure that the start and termination times for the passage times $P_j^l = T_j(l) - S_{j-1}(l)$ strictly alternate. Denote the successive passage times $P_1^1, P_1^2, \ldots, P_1^N, P_2^1, P_2^2, \ldots$ enumerated in termination order by $\{P_j' \colon j \geq 1\}$. Set $T_0' = 0$ and let $T_j'$ be the termination time for $P_j'$, $j \geq 1$.

Proposition (4.1) gives conditions which ensure that $\{(X(T_n'), P_{n+1}') \colon n \geq 0\}$ is a regenerative process in discrete time and that the expected time between regeneration points is finite. The regenerative structure guarantees (Miller [13]) that $P_n' \Rightarrow P$ as $n \to \infty$. The goal of the simulation is the estimation of $r(f) = E\{f(P)\}$, where $f$ is a real-valued (measurable) function and $P$ is the limiting passage time.

We postulate the existence of a transition $e^*$ and markings $s_0 \in B_1$ and $s_0' \in B_2$ such that a passage time terminates when transition $e^*$ fires and the marking changes from $s_0$ to $s_0'$. In addition, we assume that no passage times are underway when the marking of the SPN is $s_0'$. Formally, denote by $L(t)$ the last marking of the SPN before jumping to $X(t)$ and set

$$V(t) = (L(t), X(t)).$$

Denote by $G$ the state space of $\{V(t) \colon t \geq 0\}$. Set $A^l = A_1^l \times A_2^l$ and $B^l = B_1^l \times B_2^l$, $l = 1, 2, \ldots, N$. Now set

$$H_1^l = \left\{ s' \in S \colon (s, s') \in B^l - A^l \text{ for some } s \in S \right\}$$

## 4. PASSAGE TIMES IN STOCHASTIC PETRI NETS

Formal specification of passage times in a symmetric SPN is by means of four subsets ($A_1$, $A_2$, $B_1$, and $B_2$) of the marking set, $S$. The sets $A_1$, $A_2$, $B_1$, and $B_2$ in effect determine when to start and stop the clock measuring a particular passage time; cf. Iglehart and Shedler [7].

Denoting the jump times of the process $\{X(t):t \geq 0\}$ by $\{\zeta_n:n \geq 0\}$, for $k,n \geq 1$ we require that the sets $A_1$, $A_2$, $B_1$, and $B_2$ satisfy:

if $X(\zeta_{n-1}) \in A_1$, $X(\zeta_n) \in A_2$, $X(\zeta_{n-1+k}) \in A_1$, and $X(\zeta_{n+k}) \in A_2$
then $X(\zeta_{n-1+m}) \in B_1$ and $X(\zeta_{n+m}) \in B_2$ for some $0 < m \leq k$;

and

if $X(\zeta_{n-1}) \in B_1$, $X(\zeta_n) \in B_2$, $X(\zeta_{n-1+k}) \in B_1$, and $X(\zeta_{n+k}) \in B_2$
then $X(\zeta_{n-1+m}) \in A_1$ and $X(\zeta_{n+m}) \in A_2$ for some $0 \leq m < k$.

These conditions ensure that the start and termination times for the specified passage time strictly alternate.

In terms of the sets $A_1$, $A_2$, $B_1$, and $B_2$, define $A_1^l = \{\phi_S^l(s): s \in A_1\}$, $A_2^l = \{\phi_S^l(s): s \in A_2\}$, $B_1^l = \{\phi_S^l(s): s \in B_1\}$, and $B_2^l = \{\phi_S^l(s): s \in B_2\}$, $l = 1,2,\ldots,N$. (Recall that for $s \in S$, $\phi_S^1(s) = \phi_S(s)$ and $\phi_S^l(s) = \phi(\phi_S^{l-1}(s))$, $l = 1,2,\ldots,N$. Also recall that for $e \in E$, $\phi_E^1(e) = \phi_E(e)$ and $\phi_E^l(e) = \phi_E(\phi_E^{l-1}(e))$.) Then define two sequences of random times $\{S_j(l):j \geq 0\}$ and $\{T_j(l):j \geq 1\}$: $S_{j-1}(l)$ is the start time for the $j$th passage time (corresponding to the sets $A_1^l$, $A_2^l$, $B_1^l$, and $B_2^l$) and $T_j(l)$ is the termination time of this $j$th passage time. Set

$$S_0(l) = 0,$$

$$S_j(l) = \inf \left\{\zeta_n \geq T_j(l): X(\zeta_n) \in A_2^l,\ X(\zeta_{n-1}) \in A_1^l\right\},\ j \geq 1$$

Standard arguments establish a ratio formula for $r(f)$.

(3.15) PROPOSITION. Provided that $E\{\tau_1\} < \infty$ and $E\{\,|f(X)|\,\} < \infty$,

$$r(f) = E\{f(X)\} = \frac{E\{Y_1(f)\}}{E\{\tau_1\}}.$$

With these results Equations (2.22) and (2.23) provide point estimates and confidence intervals for $r(f)$.

(3.16) EXAMPLE. In the token ring model of Example (2.18), take $\phi(j) = j + 1$, $j = 1,2,\ldots,N$. Set

$$S' = \{s \in S: s_{4j-1} = 1 \text{ for some } j,\ j = 1,2,\ldots,N\}$$

and consider the function $f$ defined by

$$f(s) = 1_{\{S'\}}(s)$$

for $s \in S$. According to this definition, $r(f)$ is the steady state throughput of the token ring. Note that the function $f$ satisfies Equation (3.1) since (for each $l$) $s_{4\phi^l(j)-1} = 1$ if and only if $s_{4(j+l)-1} = 1$, $j = 1,2,\ldots,N$. Arguments given in [4] show that the conditions of Proposition (2.19) hold provided that the packet interarrival time random variables, $A_j$, have new better than used distributions and satisfy a positivity condition: $P\{A_j \le R_N\} > 0$, $j = 1,2,\ldots,N$.

$$= P_l\{\delta_1 = n + 1, f(X(\zeta_n)) = f(x_n), \zeta_n \le z_n,$$

$$f(X(\zeta_{n-1})) = f(x_{n-1}), \zeta_{n-1} \le z_{n-1}, \ldots, \zeta_1 \le z_1\} \; P_1\{X(\zeta_{\gamma_k}) = \phi_S^l(s_0')\}$$

for all $z_1, \ldots, z_n \ge 0$, $x_1, \ldots, x_n \in S$, and $n \ge 1$. Applying Equation (3.14) and then Equation (3.13), it follows that for all $z_1, \ldots, z_n \ge 0$ and $x_1, \ldots, x_n \in S$:

$$P_1\{\delta_{k+1} = n + 1, f(X(\zeta_{\gamma_k+n})) = f(x_n), \zeta_{\gamma\ +n} \le z_n,$$

$$f(X(\zeta_{\gamma_k+n-1})) = f(x_{n-1}), \zeta_{\gamma_k+n-1} \le z_{n-1}, \ldots, \zeta_{\gamma_k+1} \le z_1\}$$

$$= \sum_{l=1}^{N} P_l\{\delta_1 = n + 1, f(X(\zeta_n)) = f(x_n), \zeta_n \le z_n,$$

$$f(X(\zeta_{n-1})) = f(x_{n-1}), \zeta_{n-1} \le z_{n-1}, \ldots, \zeta_1 \le z_1\} \; P_1\{X(\zeta_{\gamma_k}) = \phi_S^l(s_0')\}$$

$$= \sum_{l=1}^{N} P_1\{\delta_1 = n + 1, f(X(\zeta_n)) = f(x_n), \zeta_n \le z_n,$$

$$f(X(\zeta_{n-1})) = f(x_{n-1}), \zeta_{n-1} \le z_{n-1}, \ldots, \zeta_1 \le z_1\} \; P_1\{X(\zeta_{\gamma_k}) = \phi_S^l(s_0')\}$$

$$= P_1\{\delta_1 = n + 1, f(X(\zeta_n)) = f(x_n), \zeta_n \le z_n, f(X(\zeta_{n-1})) = f(x_{n-1}), \zeta_{n-1} \le z_{n-1}, \ldots, \zeta_1 \le z_1\}$$

so that (using Equation (3.10)) the pairs of random variables $\{(Y_k(f), \tau_k) : k \ge 1\}$ are identically distributed. □

marking is $\phi_S^l(s_0')$, the definition of a symmetric SPN implies that for all $z_1,\dots,z_n \ge 0$, $x_1,\dots,x_n \in S$, and $e_{i_1},\dots,e_{i_n} \in E$ with $p(s_k;s_{k-1},e_{i_k}) > 0$:

$$P_1\{X(\zeta_n) = \phi_S^1(x_n),\ \zeta_n \le z_n,\ e_n^* = \phi_E^1(e_{i_n}),$$

$$X(\zeta_{n-1}) = \phi_S^1(x_{n-1}),\ \zeta_{n-1} \le z_{n-1}, e_{n-1}^* = \phi_E^1(e_{i_{n-1}}),\dots,e_1^* = \phi_E^1(e_{i_1})\}$$

$$= P_l\{X(\zeta_n) = \phi_S^l(x_n),\ \zeta_n \le z_n,\ e_n^* = \phi_E^l(e_{i_n}),$$

$$X(\zeta_{n-1}) = \phi_S^l(x_{n-1}),\ \zeta_{n-1} \le z_{n-1}, e_{n-1}^* = \phi_E^l(e_{i_{n-1}}),\dots,e_1^* = \phi_E^l(e_{i_1})\},$$

$l = 1,2,\dots,N$. Hence, by Equation (3.1),

(3.13) $$P_1\{\delta_1 = n+1,\ f(X(\zeta_n)) = f(x_n),\ \zeta_n \le z_n,$$

$$f(X(\zeta_{n-1})) = f(x_{n-1}),\ \zeta_{n-1} \le z_{n-1},\dots,\zeta_1 \le z_1\}$$

$$= P_l\{\delta_1 = n+1,\ f(X(\zeta_n)) = f(x_n),\ \zeta_n \le z_n,$$

$$f(X(\zeta_{n-1})) = f(x_{n-1}),\ \zeta_{n-1} \le z_{n-1},\dots,\zeta_1 \le z_1\}$$

for all $z_1,\dots,z_n \ge 0$, $x_1,\dots,x_n \in S$, and $n \ge 1$. But by the independence argument of the first part of the proof

(3.14) $$P_1\{\delta_{k+1} = n+1,\ f(X(\zeta_{\gamma_k+n})) = f(x_n),\ \zeta_{\gamma_k+n} \le z_n,$$

$$f(X(\zeta_{\gamma_k+n-1})) = f(x_{n-1}),\ \zeta_{\gamma_k+n-1} \le z_{n-1},\dots,\zeta_{\gamma_k+1} \le z_1,\ X(\zeta_{\gamma_k}) = \phi_S^l(s_0')\}$$

$\tau_{k+1} = T_{\beta_{k+1}} - T_{\beta_k}$ and the finite dimensional distributions of $\{f(X(t)):t \geq T_{\beta_k}\}$. Observe that the joint distribution of $X(T_{\beta_k}) = \phi_S^l(s_0')$ and the clocks set or reset at time $T_{\beta_k}$ depends on the past history of the SPN only through the new marking $\phi_S^l(s_0')$, the previous marking $\phi_S^l(s_0)$, and the trigger transition $\phi_E^l(e^*)$; this implies that the cycle length $\tau_{k+1}$ and $\{f(X(t)):t \geq T_{\beta_k}\}$ are independent of $\{(Y_j(f),\tau_j):j \leq k\}$. It follows that the pairs of random variables $\{(Y_k(f),\tau_k):k \geq 1\}$ are mutually independent.

Next observe that

$$Y_k(f) = \sum_{m=\gamma_{k-1}}^{\gamma_k - 1} f(X(\zeta_m))\,[\zeta_{m+1} - \zeta_m], \tag{3.10}$$

where $\zeta_{\gamma_k} = T_{\beta_k}$, $k \geq 0$. Set $\delta_k = \gamma_k - \gamma_{k-1}$, $k \geq 1$. It is sufficient to show that for all $z_1,\ldots,z_n \geq 0$, $x_1,\ldots,x_n \in S$, and $n \geq 1$:

$$P_1\{\delta_1 = n+1, f(X(\zeta_n)) = f(x_n), \zeta_n \leq z_n, f(X(\zeta_{n-1})) = f(x_{n-1}), \zeta_{n-1} \leq z_{n-1},\ldots,\zeta_1 \leq z_1\}$$

$$= P_1\{\delta_{k+1} = n+1, f(X(\zeta_{\gamma_k+n})) = f(x_n), \zeta_{\gamma_k+n} \leq z_n, f(X(\zeta_{\gamma_k+n-1})) = f(x_{n-1}), \tag{3.11}$$

$$\zeta_{\gamma_k+n-1} \leq z_{\gamma_k+n-1},\ldots,\zeta_{\gamma_k+1} \leq z_1\},$$

$k \geq 1$. Here $P_1\{\cdot\}$ denotes the conditional probability associated with starting the SPN with marking $\phi_S^1(s_0')$ and all active clocks reset at time $t = 0$ according to the distributions

$$P\{C_{0,i} \leq c\} = F(c;\phi_S^1(s_0'),e_i,\phi_S^1(s_0),\phi_E^1(e^*)) \tag{3.12}$$

for $c \geq 0$, $e_i \in E(\phi_S^1(s_0'))$. Recall that $C_{0,i}$ is the clock reading associated with transition $e_i$ at time 0. Denoting by $P_l\{\cdot\}$ the corresponding conditional probability when the initial

$$= P\{X(T_n^l) = \phi_S^l(s_0') \mid X(T_{n-1}^l) = \phi_S^l(x_{n-1}),\dots,X(T_1^l) = \phi_S^l(x_1)\},$$

$l = 1,2,\dots,N$. The result follows from Equation (3.3). □

Carry out the simulation of $\{X(t):t \geq 0\}$ in random blocks defined by the successive random times $\{T_{\beta_k}:k \geq 0\}$, defined by

$$T_{\beta_k} = \inf\{T_n^l > T_{\beta_{k-1}} : X(T_n^l) = \phi_S^l(s_0') \text{ for some } l,\ l = 1,2,\dots,N\}, \tag{3.8}$$

$k \geq 1$; $\beta_0 = 0$ and $T_{\beta_0} = 0$. (Note that the random times $\{T_{\beta_k}:k \geq 0\}$ do *not* form a sequence of regeneration points for the process $\{X(t):t \geq 0\}$.)

Set

$$\tau_k = T_{\beta_k} - T_{\beta_{k-1}}$$

and

$$Y_k(f) = \int_{T_{\beta_{k-1}}}^{T_{\beta_k}} f(X(s))ds,$$

$k \geq 1$.

(3.9) PROPOSITION. The sequence of pairs of random variables $\{(Y_k(f),\tau_k):k \geq 1\}$ are independent and identically distributed.

*Proof:* The sequence $\{\beta_k:k \geq 1\}$ are indices of the successive stopping times $\{T_n:n \geq 1\}$ at which transition $\phi_E^l(e^*)$ fires and the marking changes from $\phi_S^l(s^*)$ to $\phi_S^l(s_0')$ for some $s^* \in S^*$ and some $l$, $l = 1,2,\dots,N$. Thus, by the definition of a symmetric SPN, each of the clocks running at time $T_{\beta_k}+$ was set or can be viewed as having been probabilistically reset at time $T_{\beta_k}$. Therefore $\{X(t):t \geq T_{\beta_k}\}$ determines the distribution of

$$(3.5)\qquad P\{X(\zeta_n) = \phi_S^1(x_n),\ e_n^* = \phi_E^1(e_{i_n}),\ldots,X(\zeta_1) = \phi_S^1(x_1),\ e_1^* = \phi_E^1(e_{i_1}),\ X(\zeta_0) = \phi_S^1(x_0)\}$$

$$= P\{X(\zeta_n) = \phi_S^l(x_n),\ e_n^* = \phi_E^l(e_{i_n}),\ldots,X(\zeta_1) = \phi_S^l(x_1),\ e_1^* = \phi_E^l(e_{i_1}),\ X(\zeta_0) = \phi_S^l(x_0)\},$$

$l = 1,2,\ldots,N$. Denote by $\{\gamma_n^l : n \ge 0\}$ the indices of the successive times $\{\zeta_n : n \ge 0\}$ at which transition $\phi_E^l(e^*)$ fires when the marking is $\phi_S^l(s^*)$ for some $s^* \in S^*$. Equation (3.5) implies that for all $x_1,\ldots,x_n \in S$ and $s_1^*,\ldots,s_n^* \in S^*$:

$$(3.6)\qquad P\{X(\zeta_{\gamma_n^1}) = \phi_S^1(x_n),\ e_n^* = \phi_E^1(e^*),\ X(\zeta_{\gamma_n^1-1}) = \phi_S^1(s_n^*),\ldots,X(\zeta_{\gamma_1^1}) = \phi_S^1(x_1),$$

$$e_1^* = \phi_E^1(e^*),\ X(\zeta_{\gamma_1^1-1}) = \phi_S^1(s_1^*)\}$$

$$= P\{X(\zeta_{\gamma_n^l}) = \phi_S^l(x_n),\ e_n^* = \phi_E^l(e^*),\ X(\zeta_{\gamma_n^l-1}) = \phi_S^l(s_n^*),\ldots,X(\zeta_{\gamma_1^l}) = \phi_S^l(x_1),$$

$$e_1^* = \phi_E^l(e^*),\ X(\zeta_{\gamma_1^l-1}) = \phi_S^l(s_1^*)\},$$

$l = 1,2,\ldots,N$. Using the definition of $\{T_n^l : n \ge 0\}$, Equation (3.6) implies that for all $x_1,\ldots,x_n \in S$:

$$(3.7)\qquad P\{X(T_n^1) = \phi_S^1(x_n),\ \ldots,X(T_1^1) = \phi_S^1(x_1)\} = P\{X(T_n^l) = \phi_S^l(x_n),\ \ldots,X(T_1^l) = \phi_S^l(x_1)\}$$

$l = 1,2,\ldots,N$. Applying Equation (3.7) it follows that for all $x_1,\ldots,x_{n-1} \in S$:

$$P\{X(T_n^1) = \phi_S^1(s_0') \mid X(T_{n-1}^1) = \phi_S^1(x_{n-1}),\ldots,X(T_1^1) = \phi_S^1(x_1)\}$$

$$= \frac{P\{X(T_n^1) = \phi_S^1(s_0'),\ X(T_{n-1}^1) = \phi_S^1(x_{n-1}),\ldots,X(T_1^1) = \phi_S^1(x_1)\}}{P\{X(T_{n-1}^1) = \phi_S^1(x_{n-1})\ldots,X(T_1^1) = \phi_S^1(x_1)\}}$$

$$= \frac{P\{X(T_n^l) = \phi_S^l(s_0'),\ X(T_{n-1}^l) = \phi_S^l(x_{n-1}),\ldots,X(T_1^l) = \phi_S^l(x_1)\}}{P\{X(T_{n-1}^l) = \phi_S^l(x_{n-1})\ldots,X(T_1^l) = \phi_S^l(x_1)\}}$$

## 3. STEADY STATE ESTIMATION FOR SYMMETRIC STOCHASTIC PETRI NETS

In this section we consider estimation of $r(f)$ under the assumption that the function $f$ is symmetric in the sense that

(3.1) $$f(s) = f(\phi_S^l(s))$$

for all $s \in S$ and all $l = 1,2,\ldots,N$. Symmetry of the underlying SPN implies that regenerative cycles defined by the times at which the transition $\phi_E^1(e^*)$ fires and the marking changes to $\phi_S^1(s_0')$ can be decomposed into independent, nonidentically distributed blocks. These blocks are defined by the successive times $T_n$ at which transition $\phi_E^l(e^*)$ fires and the marking changes from $\phi_S^l(s^*)$ to $\phi_S^l(s_0')$ for some $s^* \in S^*$ and some $l$, $l = 1,2,\ldots,N$. Estimates for $r(f)$ can be based on observation of these blocks. Proposition (3.2) provides conditions which ensure that (for each $l$) transition $\phi_E^l(e^*)$ fires and the marking changes to $\phi_S^l(s_0')$ infinitely often with probability one.

Denote by $\{T_n : n \ge 1\}$ the times $T_1^1, T_1^2, \ldots, T_1^N, T_2^1, \ldots$ in increasing order.

(3.2) PROPOSITION. Suppose there exists $\delta > 0$ such that

(3.3) $$P\left\{X(T_n^1) = \phi_S^1(s_0') \mid X(T_{n-1}^1), \ldots, X(T_1^1)\right\} \ge \delta \text{ a.s..}$$

Then $P\{X(T_n) = \phi_S^l(s_0') \text{ i.o.}\} = 1$ for all $l = 1,2,\ldots,N$.

*Proof:* By Lemma 4 of [6] it suffices to show that

(3.4) $$P\left\{X(T_n^l) = \phi_S^l(s_0') \mid X(T_{n-1}^l), \ldots, X(T_1^l)\right\} \ge \delta \text{ a.s.}$$

Let $e_n^*$ denote the transition that fires at time $\zeta_n$, $n \ge 0$. The definition of a symmetric SPN implies that for all $x_0, x_1, \ldots, x_n \in S$ and $e_{i_1}, \ldots, e_{i_n} \in E$:

consistent point estimate

$$\hat{r}(n) = \frac{\bar{Y}(n)}{\bar{\tau}(n)} \tag{2.22}$$

and asymptotic $100(1-2\gamma)\%$ confidence interval

$$\hat{I}(n) = \left[\hat{r}(n) - \frac{z_{1-\gamma}\, s(n)}{\bar{\tau}(n)\, n^{1/2}}, \hat{r}(n) + \frac{z_{1-\gamma}\, s(n)}{\bar{\tau}(n)\, n^{1/2}}\right] \tag{2.23}$$

for $r(f)$. In Equation (2.22)

$$\bar{Y}(n) = n^{-1} \sum_{m=1}^{n} Y_m(f)$$

and

$$\bar{\tau}(n) = n^{-1} \sum_{m=1}^{n} \tau_m,$$

where $\tau_m$ is the length of the $m$th cycle and $Y_m(f)$ is the integral of $f(X(\cdot))$ over the $m$th cycle. The quantity $s^2(n)$ is a strongly consistent point estimate for

$$\sigma^2(f) = \text{var}\,(Y_1(f) - r(f)\tau_1)$$

and $z_{1-\gamma} = \Phi^{-1}(1-\gamma)$, where $\Phi$ is the distribution function of a standardized normal random variable, $N(0,1)$. Confidence intervals are based on the central limit theorem (c.l.t.)

$$\frac{n^{1/2}\left\{\hat{r}(n) - r(f)\right\}}{\sigma(f)/E\{\tau_1\}} \Rightarrow N(0,1)$$

as $n \to \infty$.

Also suppose that there exists $s \in S$ such that for all $s^* \in S^*$,

(i) the set $O(s_0';s^*,e^*) = E(s_0') \cap (E(s^*) - \{e^*\}) = \emptyset$,

(ii) the set $N(s_0';s^*,e^*) = E(s_0') - (E(s^*) - \{e^*\}) = N(s_0';s,e^*)$, and

(iii) the clock setting distribution $F(\cdot;s_0',e',s^*,e^*) = F(\cdot;s_0',e',s,e^*)$ for all $e' \in N(s_0';s,e^*)$.

Then $\{X(t):t \geq 0\}$ is a regenerative process in continuous time. Moreover, if

$$E\left\{T_{n+1}^1 - T_n^1\right\} \leq c < \infty \tag{2.21}$$

for all $n \geq 0$ then the expected time between regeneration points is finite.

Equation (2.20) implies that transition $\phi_E^1(e^*)$ triggers a marking change to $\phi_S^1(s_0')$ infinitely often with probability one. Furthermore, at such a time $T_n^1$, the only clocks that are active have just been set since $O(\phi_S^1(s_0');\phi_S^1(s^*),\phi_E^1(e^*)) = \emptyset$ for all $s^* \in S^*$. The *joint distribution of* $X(T_n^1)$ *and the clocks set at time* $T_n^1$ *depends on the past history of* $\{X(t):t \geq 0\}$ only through $\phi_S^1(s_0')$, the previous marking $\phi_S^1(s^*)$, and the trigger transition $\phi_E^1(e^*)$. Since the new transitions and clock setting distributions are the same for all $s^*$, the process $\{X(t):t \geq 0\}$ probabilistically restarts whenever $\{X(T_n^1):n \geq 1\}$ hits $\phi_S^1(s_0')$. Note that the result of Proposition (2.19) also holds if condition (i) is replaced by:

(i') $O(s_0';s_0,e^*) \neq \emptyset$ and for any $e' \in O(s_0';s_0,e^*)$ the clock setting distribution $F(\cdot;s',e',s,e)$ is exponential with mean which is independent of $s$, $s'$, and $e$. (Assumption (i') ensures that no matter when the clock for transition $e' \in O(s_0';s_0,e^*)$ was set, the remaining time until transition $e'$ triggers a marking change is exponentially distributed with the same mean.)

Under the conditions of Proposition (2.19), the basic limit theorem for regenerative processes asserts that $X(t) \Rightarrow X$ as $t \to \infty$. The goal of the simulation is the estimation of $r(f) = E\{f(X)\}$, where $f$ is a real-valued (measurable) function having domain $S$. From $n$ cycles the standard regenerative method (Crane and Iglehart [2]) provides the strongly

$L_1, L_2, \ldots, L_N$ are identically distributed. Under these assumptions the SPN is symmetric. For example, take $\phi(j) = j + 1$, $j = 1,2,\ldots,N$. Then

$$\phi_D(4(j-1)+k) = 4j + k,$$

$$\phi_E(e_{3(j-1)+k}) = e_{3j+k},$$

and

$$\phi_S(s_1, s_2, \ldots, s_{4N}) = (s_5, s_6, \ldots, s_{4N}, s_1, \ldots, s_4).$$

Proposition (2.19) (cf. Proposition (4.7) of [4]) gives a set of conditions on the building blocks of an SPN which ensure that the process is regenerative and that the expected time between regeneration points is finite. Set $\phi_S^1(s) = \phi_S(s)$ and

$$\phi_S^l(s) = \phi_S(\phi_S^{l-1}(s))$$

for $s \in S$ and $l = 2,\ldots,N$. Similarly, set $\phi_E^1(e) = \phi_E(e)$ and

$$\phi_E^l(e) = \phi_E(\phi_E^{l-1}(e))$$

for $e \in E$. Recall that $\zeta_n$ is the $n$th transition firing time, $n \ge 0$. Let $\{T_n^l : n \ge 0\}$ be an increasing sequence of stopping times that are finite ($T_n^l < \infty$ a.s.) transition firing times such that for some $e^* \in E$ and $S^* \subseteq S$: $T_0^l = 0$ and

$$T_n^l = \inf\left\{t > T_{n-1}^l: \text{at time } t \text{ transition } \phi_E^l(e^*) \text{ fires and the marking is } \phi_S^l(s^*) \text{ for some } s^* \in S^*\right\},$$

$n \ge 1$ and $l = 1,2,\ldots,N$.

(2.19) PROPOSITION. Suppose that there exists $s_0' \in S$ and $\delta > 0$ such that

$$P\left\{X(T_n^1) = \phi_S^1(s_0') \mid X(T_{n-1}^1),\ldots,X(T_1^1)\right\} \ge \delta \text{ a.s.} \qquad (2.20)$$

Let $\phi$ be a cyclic permutation of the set $\{1,2,...,N\}$. In terms of this permutation define a mapping, $\phi_D$, of $D$ onto $D$:

(2.14) $$\phi_D((j-1)L_1 + k) = (\phi(j) - 1)L_1 + k,$$

$j = 1,2,...,N$ and $k = 1,2,...,L_1$. Similarly, define a mapping, $\phi_E$, of $E$ onto $E$:

(2.15) $$\phi_E(e_{(j-1)M_1+k}) = e_{(\phi(j)-1)M_1+k},$$

$j = 1,2,...,N$ and $k = 1,2,...,M_1$. Also define a mapping, $\phi_S$, of $S$ onto $S$:

(2.16) $$\phi_S(s_1,s_2,...,s_L) = (s_{\phi_D(1)},s_{\phi_D(2)},...,s_{\phi_D(L)}).$$

For $D' \subseteq D$, we write $\phi_D(D') = \{\phi_D(i): i \in D'\}$ and for $E' \subseteq E$ we write $\phi_E(E') = \{\phi_E(e): e \in E'\}$.

(2.17) DEFINITION. An SPN $\{X(t):t \geq 0\}$ is said to be *symmetric* if there exists a cyclic permutation, $\phi$, of the set $\{1,2,...,N\}$ such that:

(i) $\phi_D(L(e)) = L(\phi_E(e))$, $\phi_D(I(e)) = I(\phi_E(e))$, and $\phi_D(J(e)) = J(\phi_E(e))$ for all $e \in E$,

(ii) $p(s';s,e) = p(\phi_S(s');\phi_S(s),\phi_E(e))$ for all $e \in E$ and $s,s' \in S$, and

(iii) $F(\cdot;s',e',s,e) = F(\cdot;\phi_S(s'),\phi_E(e'),\phi_S(s),\phi_E(e))$ for all $e' \in N(s';s,e)$, $e \in E$, and $s,s' \in S$.

Condition (i) ensures that the induced mappings $\phi_D$ and $\phi_E$ preserve the sets of normal input places, inhibitor input places, and output places and thus (using Equation (2.1)) $\phi_E(E(s)) = E(\phi_S(s))$ for all $s \in S$. Conditions (ii) and (iii) ensure that the mappings $\phi_S$ and $\phi_E$ preserve the new marking probabilities and the clock setting distributions.

(2.18) EXAMPLE. In the token ring model of Example (2.9), let $N$ be the number of ports so that $L_1 = 4$ and $M_1 = 3$. Suppose that (i) $R_1 = R_2 = ... = R_N$; (ii) the random variables $A_1,A_2,...,A_N$ are identically distributed; and (iii) the random variables

and when

$$s = (s_1,\ldots,s_{4(j-1)-1},1,0,1,0,0,s_{4j+1},\ldots,s_{4N}) \in S \text{ and } s' = (s_1,\ldots,s_{4(j-1)-1},0,0,1,0,1,s_{4j+1},\ldots,s_{4N}).$$

All other new marking probabilities $p(s';s,e)$ are equal to zero.

Note that when transition $e_{3j}$ fires, a token is removed from place $4(j-1)$ and a token is deposited either in place $4j-1$ or in place $4j$, depending upon whether $(s_{4j-3},s_{4j-2})$ equals (1,0) or (0,1). All other transitions are *input-deterministic* (in that exactly one token is removed from each input place when the transition fires) and *output-deterministic* (exactly one token is deposited in each output place when the transition fires).

The distribution functions of new clock times for transitions $e' \in N(s';s,e^*)$ are as follows. If $e' = e_{3j-2}$ = "arrival of packet for transmission by port $j$," then the distribution function $F(x;s',e',s,e^*) = P\{A_j \le x\}$. If $e' = e_{3j-1}$ = "end of transmission by port $j$," then the distribution function $F(x;s',e',s,e^*) = P\{L_j \le x\}$. If $e' = e_{3j}$ = "observation of token by port $j$," then the distribution function $F(x;s',e',s,e^*) = 1_{[R_{j-1},\infty)}(x)$.

We now define a symmetric SPN. Informally, an SPN is symmetric if there are mappings of places onto places, markings onto markings, and transitions onto transitions which preserve the sets $E(s)$ of enabled transitions, the new marking probabilities $p(s';s,e)$, the sets $N(s';s,e^*)$ of new transitions, and the clock setting distributions $F(\cdot;s',e',s,e^*)$. Let $\{X(t):t \ge 0\}$ be an SPN with finite marking set, $S$, and transition set, $E$. Throughout this section we assume that $D = \{1,2,\ldots,L\}$ is the index set of places and $E = \{e_1,e_2,\ldots,e_M\}$ is the transition set, where $L = L_1N$ and $M = M_1N$ for some $N \ge 2$. (We assume that all clock setting distributions have finite mean.)

so that $H_1^l$ is the set of all possible markings when a passage time $P_n^l$ terminates. Also set

$$H_2^l = \left\{s' \in S\text{: for all } s \in S, (s,s') \in G - (B^l \cup A^l) \text{ and } (s_1',s_2') \overset{A^l}{\sim} (s,s'),\right.$$

$$\left.(s,s') \overset{A^l}{\sim} (s_1'',s_2'') \text{ for some } (s_1',s_2') \in B^l,\ (s_1'',s_2'') \in A^l\right\}$$

so that $H_2^l$ is the set of all possible markings when a passage time $P_n^l$ is not underway. (For $(s,s'),(\bar{s},\bar{s}') \in G$ we write $(s,s') \overset{A^l}{\sim} (s,\bar{s}')$ if there exists a finite sequence $e_0',s_1',e_1',s_2',\ldots,s_n',e_n'$ of transitions and markings such that

$$p(s_1';s',e_0')p(s_2';s_1',e_1')\ldots p(\bar{s};s_n',e_n') > 0$$

and $(s',s_1'),(s_n',\bar{s}),(s_j',s_{j+1}') \notin A^l$, $j = 1,2,\ldots,n-1$.) We assume that

$$B_2' = B_2 \cap (H_1^1 \cup H_2^1) \cap \ldots \cap (H_1^N \cup H_2^N) \neq \emptyset$$

and that $s_0' \in B_2'$.

As in Section 2, let $\{T_n^l : n \geq 0\}$ be an increasing sequence of stopping times that are finite ($T_n^l < \infty$ a.s.) transition firing times such that for some $e^* \in E$ and $S^* \subseteq S$: $T_0^l = 0$ and

$$T_n^l = \inf\left\{t > T_{n-1}^l\text{: at time } t \text{ transition } \phi_E^l(e^*) \text{ fires and the marking is } \phi_S^l(s^*) \text{ for some } s^* \in S^*\right\},$$

$n \geq 1$ and $l = 1,2,\ldots,N$.

(4.1) PROPOSITION. Suppose that there exists $e^* \in E$, $s_0 \in B_1$, and $s_0' \in B_2'$ such that $p(s_0';s_0,e^*) > 0$ and either (i) $O(s_0';s_0,e^*) = E(s_0') \cap (E(s_0) - \{e^*\}) = \emptyset$ or (ii) $O(s_0';s_0,e^*) \neq \emptyset$ and for any $e' \in O(s_0';s_0,e^*)$ the clock setting distribution $F(\cdot;s',e',s,e)$ is exponential, independent of $s$, $s'$, and $e$. Set $v_0^1 = (\phi_S^1(s_0),\phi_S^1(s_0'))$ and suppose there exists

$\delta > 0$ such that

$$P\{V(T_n^1) = v_0^1 \mid V(T_{n-1}^1),\ldots,V(T_1^1)\} \geq \delta \text{ a.s.}. \tag{4.2}$$

Then $\{(X(T_n'),P_{n+1}'):n \geq 0\}$ is a regenerative process in discrete time. Moreover, if

$$E\{T_{n+1}^1 - T_n^1\} \leq c < \infty$$

for all $n \geq 1$ then the expected time between regeneration points is finite.

*Proof:* Since $T_n^1 < \infty$ a.s. and $P\{V(T_n^1) = v_0^1 \mid V(T_{n-1}^1),\ldots,V(T_1^1)\} \geq \delta > 0$, Lemma 4 of [6] ensures that transition $\phi_E^1(e^*)$ fires and the marking of the SPN changes from $\phi_S^1(s_0)$ to $\phi_S^1(s_0')$ infinitely often with probability one: $P\{V(T_n^1) = v_0^1 \text{ i.o.}\} = 1$. Denote by $\{\beta_k^1:k \geq 1\}$ the indices of the successive passage times $\{P_n':n \geq 1\}$ which terminate when transition $\phi_E^1(e^*)$ fires and the marking changes from $\phi_S^1(s_0)$ to $\phi_S^1(s_0')$. Let $T_0' = \beta_0^1 = 0$. We must show that

(i) $\{\beta_k^1:k \geq 0\}$ is a renewal process in discrete time

and that for any $i_1 < i_2 < \ldots < i_m$ $(m \geq 1)$ and $k \geq 0$

(ii) $\{X(T'_{\beta_k^1+i_1}),P'_{\beta_k^1+i_1+1},\ldots,X(T'_{\beta_k^1+i_m}),P'_{\beta_k^1+i_m+1}\}$ and $\{X(T'_{i_1}),P'_{i_1+1},\ldots,X(T'_{i_m}),P'_{i_m+1}\}$ have the same distribution, and $\{X(T'_{\beta_k^1+i_1}),P'_{\beta_k^1+i_1+1},\ldots,X(T'_{\beta_k^1+i_m}),P'_{\beta_k^1+i_m+1}\}$ is independent of $\{(X(T_n'),P_{n+1}'):0 \leq n < \beta_k^1\}$.

At time $T'_{\beta_k^1}$, a passage time has just terminated with no other passage times underway. Now observe that each of the clocks running at time $T'_{\beta_k^1}+$ was set or can be viewed as having been probabilistically reset at time $T'_{\beta_k^1}$. (Assumption (ii) ensures that no matter when the clock for transition $e' \in O(\phi_S^1(s_0');\phi_S^1(s_0),\phi_E^1(e^*))$ was set, the remaining time until transition $e'$ fires is exponentially distributed with the same parameter.) Therefore

$\{X(t):t \geq T'_{\beta_k}\}$ determines the finite dimensional distributions of $X(T'_{\beta^1_k+i})$, $P'_{\beta^1_k+i+1}$ for $i \geq 0$ and the distribution of $\beta^1_{k+1} - \beta^1_k$. The joint distribution of $X(T'_{\beta^1_k})$ and the clocks set or reset at time $T'_{\beta^1_k}$ depends on the past history of the SPN only through $\phi^1_S(s'_0)$, the previous marking $\phi^1_S(s_0)$, and the trigger transition $\phi^1_E(e^*)$. This distribution is the same for all $\beta^1_k$ and therefore (i) and (ii) hold.

Proposition (4.3) of [4] implies that $\{X(t):t \geq 0\}$ is a regenerative process in continuous time and $E\{T'_{\beta^1_k+1} - T'_{\beta^1_k}\} < \infty$. It follows, since the state space of the SPN is finite and the clock setting distributions have finite mean, that $E\{\beta^1_{k+1} - \beta^1_k\} < \infty$. $\square$

Proposition (4.3) provides sufficient conditions which ensure that Equation (4.2) holds. We postulate the existence of a distinguished random time $T^+_n$ in the interval $[T^1_{n-1}, T^1_n)$ and a set $\{e^{(k)}:k \in K(v^+_n)\}$ of distinguished transitions determined by the marking, $v^+_n$, at time $T^+_n$. We make the following *sample path assumption*: $V(T^1_n) = v^1_0$ when each of the distinguished transitions occurs prior to some time $T^+_n + R_{n,k}(v^+_n)$. Proposition (4.3) asserts that the geometric trials recurrence criterion (Equation (4.6)) is satisfied if the clock setting distributions associated with the distinguished transitions are "new better than used" (NBU) and satisfy a "positivity" condition (condition (iii)) which guarantees the existence of $\delta > 0$ as in Equation (4.6). (A positive random variable $A$ is NBU if

$$P\{A > x + y \mid A > y\} \leq P\{A > x\}$$

for all $x,y \geq 0$. Note that every increasing failure rate (IFR) distribution is NBU. Also, if $A$ and $B$ are independent random variables with NBU distributions, then the distributions of $A + B$, $\min(A,B)$, and $\max(A,B)$ are NBU.)

Recall that $G$ is the state space of the process $\{V(t):t \geq 0\}$. Let $\{T_n^+:n \geq 0\}$ be a sequence of transition firing times and denote the state space of $\{V(T_n^+):n \geq 0\}$ by $G^+$. Set $\mathscr{H}(T_n^+) = \{(S_l,C_l):0 \leq l < N(T_n^+)\}$, where $N(\cdot)$ is given by Equation (2.8). Let $e^{(1)},e^{(2)},\ldots,e^{(m)} \in E$ and for $v^+ = (l^+,x^+) \in G^+$, set $E(v^+) = E(x^+)$ and

$$K(v^+) = \left\{k:\ e^{(k)} \in E(v^+)\right\}.$$

When $V(T_n^+) = v^+$, for $k \in K(v^+)$ we denote by $S_{n,k}(v^+)$ the latest time less than or equal to $T_n^+$ at which the clock associated with transition $e^{(k)}$ was set and by $A_{n,k}(v^+)$ the setting on the clock at time $S_{n,k}(v^+)$.

(4.3) PROPOSITION. Let $e^{(1)},e^{(2)},\ldots,e^{(m)} \in E$ and let $\{T_n^+:n \geq 0\}$ be a sequence of transition firing times. For $v^+ \in G^+$, let $\{R_{n,k}(v^+):k \in K(v^+)\}$, be identically distributed collections of random variables, independent of $\{A_{n,k}(V(T_n^+)):k \in K(V(T_n^+))\}$ and $\mathscr{H}(T_n^+)$. Assume that:

(i) $T_{n-1}^1 \leq T_n^+$ a.s. and for $v_0,v_1,\ldots,v_{n-1} \in G$ and $v^+ \in G^+$,

$$\begin{aligned} P\left\{V(T_n^1) = v_0^1,\ V(T_n^+) = v^+,\ V(T_{n-1}^1) = v_{n-1},\ldots,V(T_1^1) = v_0\right\} \\ \geq P\Big\{S_{n,k}(v^+) + A_{n,k}(v^+) \leq T_n^+ + R_{n,k}(v^+),\ k \in K(v^+); \\ V(T_n^+) = v^+,\ V(T_{n-1}^1) = v_{n-1},\ldots,V(T_1^1) = v_0\Big\}; \end{aligned} \tag{4.4}$$

(ii) for all $e^{(k)}$ the clock setting distribution $F(\cdot;s',e^{(k)},s,e) = F(\cdot;e^{(k)})$ and is NBU; and

(iii) there exists $\delta > 0$ such that for $v^+ \in G^+$

$$\delta(v^+) = P\left\{A_k(v^+) \leq R_{n,k}(v^+),\ k \in K(v^+)\right\} \geq \delta, \tag{4.5}$$

where the random variable $A_j(v^+)$ has distribution $F(\cdot;e^{(j)})$ and $\{A_j(v^+):j \in K(v^+)\}$ are mutually independent and independent of $\{R_{n,j}(v^+):j \in K(v^+)\}$.

Then

$$(4.6) \qquad P\{V(T_n^1) = v_0^1 \mid V(T_{n-1}^1),\dots,V(T_1^1)\} \geq \delta \text{ a.s}$$

so that $P\{V(T_n^1) = v_0^1 \text{ i.o.}\} = 1$.

Proposition (4.3) follows directly from Proposition (2.16) of [3] since the process $\{V(t):t \geq 0\}$ is a generalized semi-Markov process with state space, $G$, and event set, $E$.

(4.8) EXAMPLE. In the token ring model of Example (2.17), take $\phi(j) = j + 1$, $j = 1,2,\dots,N$. Set $s_0 = (1,0,0,0,\dots,1,0,0,0,1,0,0,1)$ and $s_0' = (1,0,1,0,1,0,0,0,\dots,1,0,0,0)$. Take $e^* = e_3$ and

$$S^* = \{(s_1,s_2,\dots,s_{4N}) \in S: s_{4N} = 1\}$$

(where $S$ is given in Equation (2.13)) so that $T_n^1$ is the $n$th time at which port 2 observes the token, $n \geq 1$. (Note that $X(T_n^1) = \phi_S^1(s_0')$ if there is a packet queued for transmission at each of the other ports and port 2 starts transmission of a packet at time $T_n^1$. The SPN $\{X(t):t \geq 0\}$ changes marking to $\phi_S^1(s_0')$ when transition $\phi_E^1(e^*)$ fires and the current marking is $\phi_S^1(s_0)$.) Observe that $T_n^1 < \infty$ a.s. since

$$E\{T_n^1 - T_{n-1}^1\} \leq N\,R_1 + N\,E\{L_1\} < \infty$$

for all $n \geq 1$. Take $e^{(j)} = e_{3j-2}$ $(m = N)$. Let $T_n^+$ be the first time after $T_{n-1}^1$ at which the control token leaves port 1 (transition $\phi_E^1(e^*)$ becomes enabled). Take $R_{n,k}(v^+) = R_1$ for all $v^+ \in G^+$. Since the SPN has marking $\phi_S^1(s_0')$ at time $T_n^1$ if each transition $e_{3j-2}$ enabled at time $T_n^+$ fires before $\phi_E^1(e_3)$ fires, condition (i) of Proposition (4.3) is satisfied. Assume that for $j = 1,2,\dots,N$: (i) the distribution of $A_j$ is NBU and (ii)

$\delta_j = P\{A_j \le R_1\} > 0$ so that

$$\delta(v^+) = \prod_{j \in K(v^+)} \delta_j \ge \prod_{j=1}^{N} \delta_j = \delta > 0.$$

Then $P\{V(T_n^1) = v_0^1 \text{ i.o.}\} = 1$.

The definition of a symmetric SPN implies that, for the process $\{(X(T_n'),P'_{n+1}):n \ge 0\}$, regenerative cycles defined by the times at which the transition $\phi_E^1(e^*)$ fires and the marking changes from $\phi_S^1(s_0)$ to $\phi_S^1(s_0')$ can be decomposed into independent, nonidentically distributed blocks. These blocks are defined by the successive times $T_n$ at which transition $\phi_E^l(e^*)$ fires and the marking changes from $\phi_S^l(s^*)$ to $\phi_S^l(s_0')$ for some $s^* \in S^*$ and some $l$, $l = 1,2,\ldots,N$. Estimates for characteristics of limiting passage times can be based on measurement of passage times contained in these blocks. Denote the state space of the process $\{V(T_n^l):n \ge 0\}$ by $G^l$ and set

$$v_0^l = (\phi_S^l(s_0),\phi_S^l(s_0')), \tag{4.8}$$

$l = 1,2,\ldots,N$.

Denote by $\{T_n:n \ge 1\}$ the times $T_1^1,T_1^2,\ldots,T_1^N,T_2^1,\ldots$ in increasing order.

(4.9) PROPOSITION. Suppose there exists $\delta > 0$ such that

$$P\left\{V(T_n^1) = v_0^1 \mid V(T_{n-1}^1),\ldots,V(T_1^1)\right\} \ge \delta \text{ a.s..} \tag{4.10}$$

Then $P\{V(T_n) = v_0^l \text{ i.o.}\} = 1$ for all $l = 1,2,\ldots,N$.

Arguments analogous to those given in Section 3 establish Proposition (4.9). Using symmetry of the SPN, the idea is to show that

$$(4.11)\qquad P\{V(T_n^1) = v_0^1 \mid V(T_{n-1}^1) = \phi_S^1(v_{n-1}),\ldots,V(T_1^1) = \phi_S^1(v_0)\}$$

$$= P\{V(T_n^l) = v_0^l \mid V(T_{n-1}^l) = \phi_S^l(v_{n-1}),\ldots,V(T_1^l) = \phi_S^l(v_0)\}$$

for all $v_0,v_1,\ldots,v_{n-1} \in G^N$. (For $v = (s,s') \in G^N$ we write $\phi_S(v) = (\phi_S(s),\phi_S(s'))$.)

Carry out the simulation of $\{V(t):t \geq 0\}$ in random blocks defined by the successive random times $\{T'_{\beta_k}:k \geq 0\}$, where

$$(4.12)\qquad T'_{\beta_k} = \inf\{T_n^l > T'_{\beta_{k-1}} : V(T_n^l) = v_0^l \text{ for some } l,\ l = 1,2,\ldots,N\},$$

$k \geq 1$; $\beta_0 = 0$ and $T'_{\beta_0} = 0$. Each epoch $T'_{\beta_k}$ corresponds to the termination of a passage time with no other passage times underway. (Note that the random times $\{T'_{\beta_k}:k \geq 0\}$ do *not* form a sequence of regeneration points for the process $\{(X(T'_n),P'_{n+1}):n \geq 0\}$.)

Set $\alpha_k = \beta_k - \beta_{k-1}$, $k \geq 1$. According to this definition $\alpha_k$ is the number of passage times in the $k$th block. Also set

$$Y_1(f) = \sum_{j=1}^{\alpha_1} f(P'_j)$$

and denote the analogous quantity in the $k$th block by $Y_k(f)$, $k \geq 1$.

(4.13) PROPOSITION. The sequence of pairs of random variables $\{(Y_k(f),\alpha_k):k \geq 1\}$ are independent and identically distributed.

*Proof:* As in the proof of Proposition (4.1), observe that at time $T'_{\beta_k}$ defined by Equation (4.12) a passage time has just terminated with no passage times underway and each of the

clocks running at time $T'_{\beta_k}+$ was set or can be viewed as having been probabilistically reset at time $T'_{\beta_k}$. Therefore $\{X(t):t \geq T'_{\beta_k}\}$ determines the distribution of $\alpha_{k+1} = \beta_{k+1} - \beta_k$ and the finite dimensional distributions of $P'_{\beta_k+i+1}$ for $i \geq 0$. The joint distribution of the clocks set or reset at time $T'_{\beta_k}$ depends on the past history of the SPN only through $X(T'_{\beta_k}) = \phi^l_S(s'_0)$, the previous marking $\phi^l_S(s_0)$, and the trigger transition $\phi^l_E(e^*)$. It follows that the pairs of random variables $\{(Y_k(f),\alpha_k):k \geq 1\}$ are mutually independent.

Recall that $\zeta_n$ is the time of the $n$th transition firing and denote by $e^*_n$ the transition that fires at time $\zeta_n$, $n \geq 0$. Also recall that $C_n$ is the vector of clock readings at time $\zeta_n$ and that $C_{n,i}$ is the $i$th coordinate of the vector $C_n$ for $e_i \in E(S_n)$. Let $z_1,\ldots,z_n \geq 0$, $x_1,\ldots,x_n \in S$ and $e_{i_1},\ldots,e_{i_n} \in E$ with $p(x_k;x_{k-1},e_{i_k}) > 0$. It follows from the definition of a symmetric SPN that

$$(4.14) \qquad P_1\{X(\zeta_n) = \phi^1_S(x_n), \zeta_n \leq z_n, e^*_n = \phi^1_E(e_{i_n}), X(\zeta_{n-1}) = \phi^1_S(x_{n-1}),$$

$$\zeta_{n-1} \leq z_{n-1}, e^*_{n-1} = \phi^1_E(e_{i_{n-1}}),\ldots, e^*_1 = \phi^1_E(e_{i_1})\}$$

$$= P_l\{X(\zeta_n) = \phi^l_S(x_n), \zeta_n \leq z_n, e^*_n = \phi^l_E(e_{i_n}), X(\zeta_{n-1}) = \phi^l_S(x_{n-1}),$$

$$\zeta_{n-1} \leq z_{n-1}, e^*_{n-1} = \phi^l_E(e_{i_{n-1}}),\ldots, e^*_1 = \phi^l_E(e_{i_1})\}$$

for all $l = 1,2,\ldots,N$. (Here $P_1\{\cdot\}$ denotes the conditional probability associated with starting the SPN with marking $\phi^1_S(s'_0)$ and all active clocks reset at time $t = 0$ according to the distributions

$$P\{C_{0,i} \leq c\} = F(c;\phi^1_S(s'_0),e_i,\phi^1_S(s_0),\phi^1_E(e^*))$$

for $c \geq 0$, $e_i \in E(\phi_S^1(s_0'))$; $P_l\{\cdot\}$ denotes the corresponding conditional probability when the initial marking is $\phi_S^l(s_0')$.)

Next suppose that $X(0) = s_0'$ and that all active clocks are reset at time $t = 0$ according to the distributions $F(c;s_0',e_i,s_0,e^*)$, $e_i \in E(s_0')$. Set $X^1(t) = \phi_S^1(X(t))$ and $X^l(t) = \phi_S^l(X(t))$, $t \geq 0$. Observe that for each sample path of $\{X(t):t \geq 0\}$ and all $n \geq 0$,

$$X^1(\zeta_{n-1}) \in A_1^{m_1} = \{\phi_S^{m_1}(s): s \in A_1\} \text{ and } X^1(\zeta_n) \in A_2^{m_1} = \{\phi_S^{m_1}(s): s \in A_2\}$$

for some $m_1$ if and only if $X^l(\zeta_{n-1}) \in A_1^{m_l}$ and $X^l(\zeta_n) \in A_2^{m_l}$ for some $m_l$. Similarly,

$$X^1(\zeta_{n-1}) \in B_1^{m_1} = \{\phi_S^{m_1}(s): s \in B_1\} \text{ and } X^1(\zeta_n) \in B_2^{m_1} = \{\phi_S^{m_1}(s): s \in B_2\}$$

for some $m_1$ if and only if $X^l(\zeta_{n-1}) \in B_1^{m_l}$ and $X^l(\zeta_n) \in B_2^{m_l}$ for some $m_l$. Since

$$S_j(m) = \inf\left\{\zeta_n \geq T_j(m): X(\zeta_n) \in A_2^m, X(\zeta_{n-1}) \in A_1^m\right\}$$

and

$$T_j(m) = \inf\left\{\zeta_n > S_{j-1}(m): X(\zeta_n) \in B_2^m, X(\zeta_{n-1}) \in B_1^m\right\}$$

for all $m$, Equation (4.14) implies that

$$(4.15) \qquad P_1\{\alpha_1 = n+1, P_{n+1}' \leq y_{n+1}, P_n' \leq y_n, \ldots, P_1' \leq y_1\}$$

$$= P_l\{\alpha_1 = n+1, P_{n+1}' \leq y_{n+1}, P_n' \leq y_n, \ldots, P_1' \leq y_1\}$$

for all $l = 1,2,\ldots,N$, $y_1,y_2,\ldots,y_{n+1} \geq 0$, and $n \geq 0$. By the independence argument in the first part of the proof it follows that

$$(4.16)\qquad P_1\{\alpha_{k+1} = n+1,\ P'_{\beta_{k+1}} \le y_{n+1},\ P'_{\beta_{k+1}-1} \le y_n,\dots,\ P'_{\beta_k+1} \le y_1,\ X(T'_{\beta_k}) = \phi^l_S(s'_0)\}$$

$$= P_l\{\alpha_1 = n+1,\ P'_{n+1} \le y_{n+1},\ P'_n \le y_n,\dots,\ P'_1 \le y_1\}\ P_1\{X(T'_{\beta_k}) = \phi^l_S(s'_0)\}$$

for all $n \ge 0$ and $l = 1,2,\dots,N$. Using Equation (4.15) this implies

$$\sum_{l=1}^{N} P_1\{\alpha_{k+1} = n+1,\ P'_{\beta_{k+1}} \le y_{n+1},\ P'_{\beta_{k+1}-1} \le y_n,\dots,\ P'_{\beta_k+1} \le y_1,\ X(T'_{\beta_k}) = \phi^l_S(s'_0)\}$$

$$= \sum_{l=1}^{N} P_l\{\alpha_1 = n+1,\ P'_{n+1} \le y_{n+1},\ P'_n \le y_n,\dots,\ P'_1 \le y_1\}\ P_1\{X(T'_{\beta_k}) = \phi^l_S(s'_0)\}$$

$$= \sum_{l=1}^{N} P_1\{\alpha_1 = n+1,\ P'_{n+1} \le y_{n+1},\ P'_n \le y_n,\dots,\ P'_1 \le y_1\}\ P_1\{X(T'_{\beta_k}) = \phi^l_S(s'_0)\}$$

so that

$$P_1\{\alpha_{k+1} = n+1,\ P'_{\beta_{k+1}} \le y_{n+1},\ P'_{\beta_{k+1}-1} \le y_n,\dots,\ P'_{\beta_k+1} \le y_1\}$$

$$= P_1\{\alpha_1 = n+1,\ P'_{n+1} \le y_{n+1},\ P'_n \le y_n,\dots,\ P'_1 \le y_1\}$$

and the pairs of random variables $\{(Y_k(f),\alpha_k):k \ge 1\}$ are identically distributed. □

Standard arguments establish a ratio formula for $r(f) = E\{f(P)\}$.

(4.17) PROPOSITION. Provided that $E\{\tau_1\} < \infty$, $P\{P \in D(f)\} = 0$ and $E\{\,|f(P)|\,\} < \infty$,

$$E\{f(P)\} = \frac{E\{Y_1(f)\}}{E\{\alpha_1\}}.$$

With these results, based on $n$ blocks (cf. Crane and Iglehart [2]) a strongly consistent point estimate for $r(f)$ is

(4.18) $$\hat{r}(n) = \frac{\overline{Y}(n)}{\overline{\alpha}(n)}$$

and an asymptotic $100(1-2\gamma)\%$ confidence interval is

(4.19) $$\hat{I}(n) = \left[\hat{r}(n) - \frac{z_{1-\gamma}\, s(n)}{\overline{\alpha}(n)\, n^{1/2}},\ \hat{r}(n) + \frac{z_{1-\gamma}\, s(n)}{\overline{\alpha}(n)\, n^{1/2}}\right],$$

where $s^2(n)$ is a strongly consistent point estimate for $\sigma^2(f) = \mathrm{var}\,(Y_1(f) - r(f)\alpha_1)$. Confidence intervals are based on the c.l.t.

(4.20) $$\frac{n^{1/2}\left\{\hat{r}(n) - r(f)\right\}}{\sigma(f)/E\{\alpha_1\}} \Rightarrow N(0,1)$$

as $n \to \infty$.

(4.21) EXAMPLE. In the token ring model of Example (2.18), consider port access times measured from the arrival of a packet for transmission by some port until the start of transmission by the port. This sequence of passage times is specified by the four subsets $A_1 = \{(s_1,\ldots,s_{4N}) \in S\colon s_1 = s_3 = 0\}$, $A_2 = \{(s_1,\ldots,s_{4N}) \in S\colon s_1 = 1 \text{ and } s_3 = 0\}$, $B_1 = \{(s_1,\ldots,s_{4N}) \in S\colon s_3 = 0 \text{ and } s_{4N} = 1\}$, and $B_2 = \{(s_1,\ldots,s_{4N}) \in S\colon s_3 = 1 \text{ and } s_{4N} = 0\}$. The set of all possible markings when a passage time $P_j^l$ terminates or is not underway is $H^l = \{(s_1,s_2,\ldots,s_{4N}) \in S\colon s_{4l-3}s_{4l-1} = 1 \text{ or } s_{4l-2} = 1\}$. Then $B_2' \neq \emptyset$ and $s_0' = (1,0,1,0,0,1,0,0,\ldots,0,1,0,0) \in B_2'$. The random times $\{T'_{\beta_k}\colon k \geq 0\}$ correspond to terminations of access times which occur when there is no packet queued for transmission at any of the ports. Propositions (4.13) and (4.17) hold provided that the packet interarrival time random variables are exponentially distributed. (The random time $T_n^l$ is

the $n$th time at which port $l+1$ observes the token, $n \geq 0$. Note that $\{V(T_n^l):n \geq 0\}$ is an irreducible, finite state discrete time Markov chain so that $P\{V(T_n^l) = v_0^l \text{ i.o.}\} = 1$. It follows that $P\{V(T_n) = v_0^l \text{ i.o.}\} = 1$ for all $l = 1,2,\ldots,N$.)

## 5. STATISTICAL EFFICIENCY

Section 4 provides two estimation procedures for passage times in a symmetric SPN. Each of these procedures rests on the assumption that there exist $e^* \in E$, $s_0 \in B_1$, and $s_0' \in B_2'$ satisfying the conditions of Proposition (4.1). The regenerative structure guarantees that $P_n' \Rightarrow P$ as $n \to \infty$ and the goal of the simulation is the estimation of $r(f) = E\{f(P)\}$, where $f$ is a real-valued measurable function. (We assume that the function $f$ is such that $E\{|f(P)|\} < \infty$ and $P\{P \in D(f)\} = 0$ so that ratio formulas for $r(f)$ hold.)

Estimates for $r(f)$ can be based on measurement of passage times $\{P_n^1:n \geq 1\}$ and simulation of the underlying SPN in regenerative cycles defined by the times $T_n^l$ at which $V(T_n^l) = v_0^1$. Alternatively, exploiting properties of a symmetric SPN, estimates can be based on measurement of passage times $\{P_n':n \geq 1\}$ and simulation of the underlying SPN in independent, nonidentically distributed blocks defined by the times $T_n^l$ at which $V(T_n^l) \in \{v_0^1,\ldots,v_0^N\}$. This estimation procedure extracts more passage time information from a simulation of fixed length and should provide estimates for $r(f)$ that are relatively more accurate. In this section we verify that this is indeed the case by showing that the resulting confidence intervals are shorter.

For $t \geq 0$ let $m^1(t)$ be the number of passage times $\{P_n^1 : n \geq 1\}$ completed in $(0,t]$ and denote by $\{\beta_k^1 : k \geq 1\}$ the indices of the successive termination times $\{T_n' : n \geq 1\}$ at which $V(T_n') = v_0^1$. Set

$$\alpha_k^1 = m^1(T'_{\beta_k^1}) - m^1(T'_{\beta_{k-1}^1}),$$

$$Y_k^1(f) = \sum_{j=m^1(T'_{\beta_{k-1}^1})+1}^{m^1(T'_{\beta_k^1})} f(P_j^1),$$

$k \geq 1$. Also set

$$(\sigma^1(f))^2 = \mathrm{var}\left(Y_1^1(f) - r(f)\alpha_1^1\right).$$

Then by Lemma (4.1) of Iglehart and Shedler [5],

(5.1)
$$\frac{t^{1/2}\left(\dfrac{1}{m^1(t)}\sum_{j=1}^{m^1(t)} f(P_j^1) - r(f)\right)}{(E\{\tau_1^1\})^{1/2}\sigma^1(f)/E\{\alpha_1^1\}} \Rightarrow N(0,1)$$

as $t \to \infty$ provided that $E\{(\alpha_1^1)^2\} < \infty$ and $E\{(Y_1^1(|f|))^2\} < \infty$. Here $\tau_k^1 = T'_{\beta_k^1} - T'_{\beta_{k-1}^1}$. Since the numerator in this c.l.t. and the limit ($N(0,1)$) is independent of the transition $\phi_E^1(e^*)$ and the markings $\phi_S^1(s_0)$ and $\phi_S^1(s_0')$ which define the cycles, so is the denominator; this is a consequence of the convergence of types theorem (Billingsley [1], Theorem 14.2). Thus, the quantity

$$e^1(f) = (E\{\tau_1^1\})^{1/2}\sigma^1(f)/E\{\alpha_1^1\}$$

is an appropriate measure of the statistical efficiency of the estimation procedure based on cycles.

Now let $m(t)$ be the number of passage times $\{P'_n : n \geq 1\}$ completed in $(0,t]$. Set

$$\alpha_k = m(T'_{\beta^1_k}) - m(T'_{\beta^1_{k-1}}),$$

$$Y_k(f) = \sum_{j=m(T'_{\beta^1_{k-1}})+1}^{m(T'_{\beta^1_k})} f(P'_j),$$

and

$$(\sigma(f))^2 = \text{var}\,(Y_1(f) - r(f)\alpha_1).$$

Again using Lemma (4.1) of [5],

$$\frac{t^{1/2}\left(\dfrac{1}{m(t)}\displaystyle\sum_{j=1}^{m(t)} f(P'_j) - r(f)\right)}{(E\{\tau^1_1\})^{1/2}\sigma(f)/E\{\alpha_1\}} \Rightarrow N(0,1) \tag{5.2}$$

as $t \to \infty$ provided that $E\{(\alpha_1)^2\} < \infty$ and $E\{(Y_1(|f|))^2\} < \infty$. Now observe that the numerator and the limit in this c.l.t. is independent of whether the passage times $\{P'_n : n \geq 1\}$ are measured in regenerative cycles (defined by transition $\phi^1_E(e^*)$ and markings $\phi^1_S(s_0)$ and $\phi^1_S(s'_0)$) or in blocks (defined by $\phi^l_E(e^*)$, $\phi^l_S(s_0)$, and $\phi^l_S(s'_0)$ for all $l = 1,2,\ldots,N$.) Therefore,

$$e(f) = (E\{\tau^1_1\})^{1/2}\sigma(f)/E\{\alpha_1\}$$

is an appropriate measure of statistical efficiency of the estimation procedure based on blocks.

Note that when the passage times $\{P_n^1 : n \geq 1\}$ are used to construct point and interval estimates for $r(f)$, the half-length of the confidence interval is proportional to $e^1(f)$, and when the passage times $\{P_n' : n \geq 1\}$ are used, (with the same constant of proportionality) the half-length of the confidence interval is proportional to $e(f)$. Proposition (5.3) asserts that under mild regularity conditions on the function $f$, $e(f) \leq e^1(f)$.

(5.3) PROPOSITION. For all functions $f$ such that $E\{|f(P)|\} < \infty$ and $P\{P \in D(f)\} = 0$, $e(f) \leq e^1(f)$.

*Proof:* It is sufficient to show that

$$(\sigma(f))^2 \leq N^2 \, (\sigma^1(f))^2 \tag{5.4}$$

and

$$E\{\alpha_1\} = N \, E\{\alpha_1^1\}. \tag{5.5}$$

To establish Equation (5.4), for $t \geq 0$ set

$$W(t) = \sum_{j=1}^{m(t)} f(P_j') - r(f) m(t).$$

Now observe that $\{X(t): t \geq 0\}$ is a regenerative process by Proposition (2.19) and that, with respect to this process, $\{W(t): t \geq 0\}$ is a cumulative process in the sense of Smith [17] with

$$E\{W(T'_{\beta_k^1}) - W(T'_{\beta_{k-1}^1})\} = E\{Y_k(f) - r(f)\alpha_k\} = 0.$$

Thus, by Theorem 8 of [17],

$$\lim_{t\to\infty} \frac{\operatorname{var}(W(t))}{t} = \frac{(\sigma(f))^2}{E\{\tau_1^1\}}. \tag{5.6}$$

Next recall that $\{P_n':n \ge 1\}$ is the sequence of passage times $P_1^1,P_1^2,\ldots,P_1^N,P_2^1,P_2^2,\ldots$ enumerated in termination order and therefore

$$\sum_{j=1}^{m(t)} f(P_j') - r(f)m(t) = \sum_{l=1}^{N} \left\{ \sum_{j=1}^{m^l(t)} f(P_j^l) - r(f)m^l(t) \right\},$$

where $m^l(t)$ is the number of passage times $\{P_n^l:n \ge 1\}$ completed in $(0,t]$. Now set

$$W^l(t) = \sum_{j=1}^{m^l(t)} f(P_j^l) - r(f)m^l(t)$$

so that

$$W(t) = \sum_{l=1}^{N} W^l(t)$$

and by the Cauchy-Schwarz inequality

$$\operatorname{var}(W(t)) \le \sum_{l=1}^{N} \operatorname{var}(W^l(t)) + \sum_{j\ne l} \{\operatorname{var}(W^j(t)) \operatorname{var}(W^l(t))\}^{1/2}$$

$$= \left[ \sum_{l=1}^{N} \{\operatorname{var}(W^l(t))\}^{1/2} \right]^2.$$

Equation (5.4) follows since

$$\lim_{t\to\infty} \frac{\operatorname{var}(W^l(t))}{t} = \frac{(\sigma^1(f))^2}{E\{\tau_1^1\}}$$

for $l = 1,2,\ldots,N$. To see this, fix $l$ and let $\{\beta_k^l : k \geq 1\}$ be the indices of the successive termination times $\{T_n^l : n \geq 1\}$ at which $V(T_n^l) = v_0^l$. Observe that $\{W^l(t) : t \geq 0\}$ is a cumulative process so that by Theorem 8 of [17],

$$\lim_{t \to \infty} \frac{\mathrm{var}\,(W^l(t))}{t} = \frac{(\sigma^l(f))^2}{E\{\tau_2^l\}},$$

where $\tau_k^l = T_{\beta_k^l}^l - T_{\beta_{k-1}^l}^l$ and

$$(\sigma^l(f))^2 = \mathrm{var}\,\left(Y_2^l(f) - r(f)\alpha_2^l\right),$$

with $\alpha_k^l = m^l(T_{\beta_k^l}^l) - m^l(T_{\beta_{k-1}^l}^l)$ and

$$Y_k^l(f) = \sum_{j=m^l(T_{\beta_{k-1}^l}^l)+1}^{m^l(T_{\beta_k^l}^l)} f(P_j^l).$$

The definition of a symmetric SPN implies that $E\{\tau_2^l\} = E\{\tau_2^1\}$ and $(\sigma^l(f))^2 = \mathrm{var}\,(Y_2^1(f) - r(f)\alpha_2^1)$.

To establish Equation (5.5) set

$$m(t) = \sum_{l=1}^{N} m^l(t)$$

and observe that $\{m^l(t) : t \geq 0\}$ and $\{m(t) : t \geq 0\}$ are cumulative processes with respect to $\{X(t) : t \geq 0\}$. Moreover,

$$\lim_{t \to \infty} \frac{E\{m(t)\}}{t} = \frac{E\{\alpha_1^1\}}{E\{\tau_1^1\}}$$

and

$$\lim_{t\to\infty} \frac{E\{m^l(t)\}}{t} = \frac{E\{\alpha_2^l\}}{E\{\tau_2^l\}}.$$

Again, since the SPN is symmetric, $E\{\tau_1^l\} = E\{\tau_2^1\}$ and $E\{\alpha_2^l\} = E\{\alpha_2^1\}$ so that

$$\lim_{t\to\infty} \frac{E\{m^l(t)\}}{t} = \frac{E\{\alpha_1^1\}}{E\{\tau_1^1\}},$$

$l = 1,2,\ldots,N.$ □

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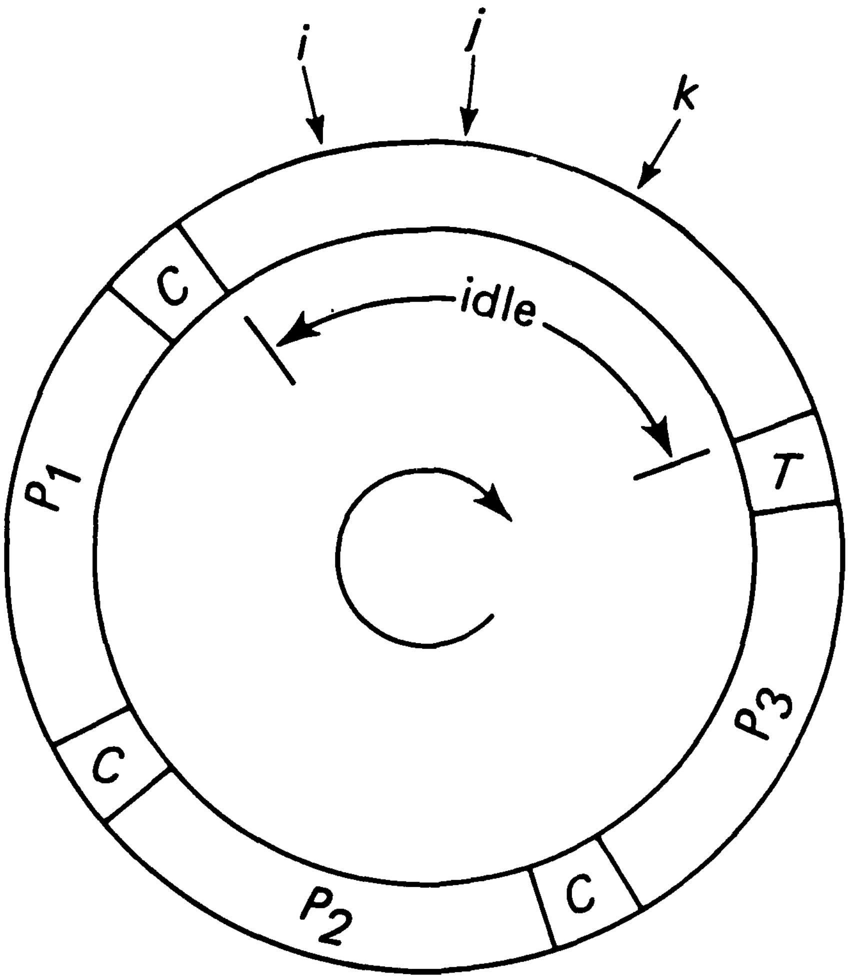


**Figure 1.** Token ring.

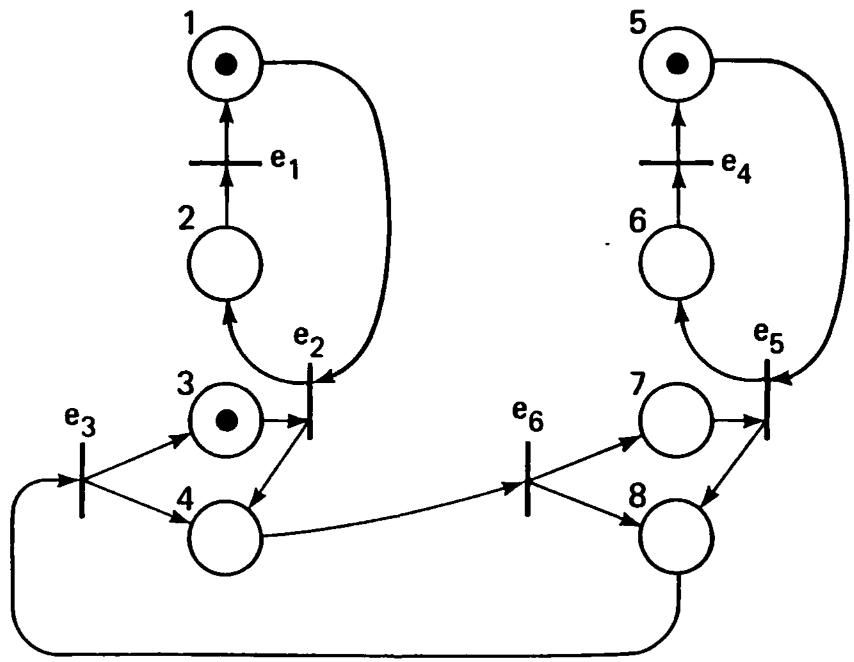


Figure 2. SPN representation of token ring.

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